III. *Propositions selected from a Paper on the Division of Right Lines, Surfaces, and Solids.* **By** James Glenie, *A. M. of the University of* Edinburgh. **Communicated** *by the* Astronomer Royal.

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PROPOSITION I. THEOREM.

If from the angles at the base of any right-lined triangle, right lines be drawn to the alternate angles of rhombi, *described upon the opposite sides, and applied reciprocally to the sides produced; and from the vertex, through the intersection of these lines, a right line be drawn to meet the base: the segments of the base, made thereby, will have to each other the duplicate proportion of the sides.*

LET ACB be any right lined triangle. Let AFEC, CDGB be rhombi, on any two sides AC, CB of this triangle, applied respectively to CB, AC, produced: from the alternate angles EFA, DGB, of which let FA, GA, be right lines drawn to the angles at the base, or third side, AB. Then, if through the intersection O of these
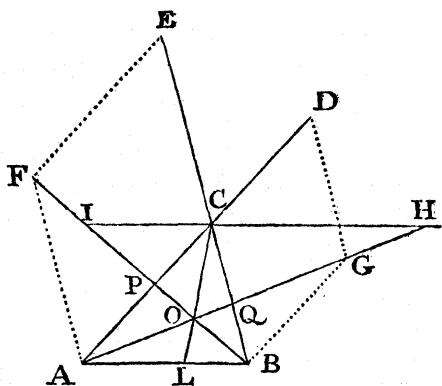
thefe lines, a right line
COL be drawn from the
vertex C to meet the
bafe AB; the fegments
AL, LB, of the bafe made
thereby, will have to
each other the dupli-
cate proportion of the
fides AC, CB. For
through the vertex C
let there be a right line
drawn parallel to AB, to meet BF, AG produced, if ne-
ceffary. Then, fince the triangles CQH, CPI, are re-
fpectively equiangular to the triangles AQB, APB (15.
and 29. E. 1.); the proportions of CH to AB and of AB
to IC are refpectively equal to the proportions of CQ
to QB and of AP to PC (4. E. 6.). But the proportion
of CH to IC is compounded of the proportion of CH to
AB, and of AB to IC; and confequently is equal to the
proportion compounded of the proportions of CQ to QB,
and of AP to PC. And, fince the triangles ACQ, APF,
are refpectively equiangular to the triangles BQG, BPC
(15. and 29. E. 1.); the proportions of CQ to QB and of
AP to PC are each equal to the proportion of AC to CB
(4. E. 6.); and when compounded are equal to the du-
plicate proportion of AC to CB. Wherefore the propor-
tion of CH to IC, which hath been fhewn to be equal to
the proportion compounded of the proportions of CQ to
QB and of AP to PC, is alfo equal to the duplicate propor-
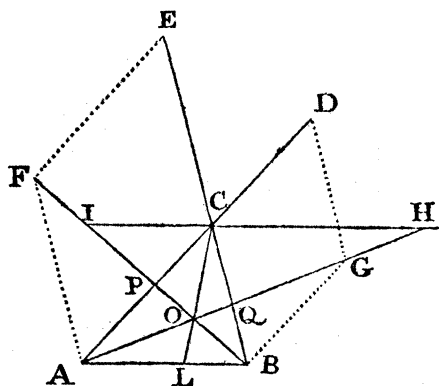
4   tion

tion of AC to CB (11. E. 5.). But, fince the triangles COH, COI, are refpectively equiangular to the triangles AOL, LOB, the proportion of CH to IC is equal to the proportion of AL to LB (4. E. 6. and 16. E. 5.). Therefore the proportion of AL to LB is equal to the duplicate proportion of AC to CB (11. E. 5.). *Q. E. D.*

COR. I. If the triangle be ifofceles, the right line drawn from the vertex to the bafe is perpendicular thereto, and the fegments of the bafe are equal to each other.

COR. II. When the triangle is right-angled, the line drawn from the vertex to the bafe is always perpendicular to it (as appears from 8. E. 6. and its cor.); and the *rhombi* become fquares on the fides comprehending the right angle.

COR. III. The fegments of the fides adjacent to the bafe, are refpectively third proportionals to the fum of the fides, and the fides themfelves.

COR. IV. The fegments of the fides adjacent to the vertex are equal to each other, and each of them is a fourth proportional to the fum of the fides, and the fides themfelves (*a*).

COR.

(*a*) And it may be added, a mean in proportion between the two fegments adjacent to the bafe. For if a right line AB be any how divided in c, and from
A   D   C   E   B
the two fegments CA, CB, third proportionals to the whole line and each fegment refpectively, CD, CE, be taken away, the remainders AD, EB, are equal, and each is a mean in proportion between the two CD, CE. For becaufe AB : AC = AC : CD;

L 2   therefore,

COR. V. The fegments of the bafe are proportional to the fegments of the fides, which are adjacent to them.
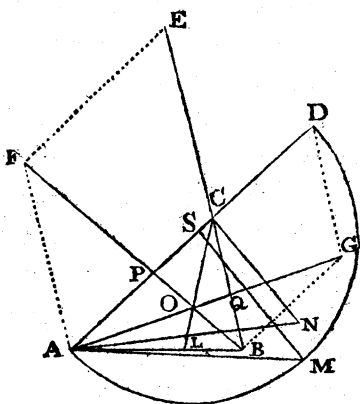
PROPOSITION II.

Let there be any two right lines given. There is an angle which may be made by thefe lines; fuch that if, from their extremities which do not meet, right lines be drawn to the alternate angles of rhombi defcribed on them, and reciprocally applied to them when produced; and from the faid angle through the interfection of thefe lines, a right line be drawn to meet the right line joining the faid extremities; the fegments of this line made thereby, fhall be refpectively equal to the adjacent fegments of the given lines.

therefore, by converfion, AB : BC � CA : AD. Again, becaufe AB : BC � BC : CE, by converfion AB : AC � BC : BE : and by permutation AB : BC � AC : BE. Therefore AC : BE � AC : AD. Therefore AD and BE are equal. I fay, moreover, that each of the two equal lines AD, BE, is a mean in proportion between the two CD, CE. For becaufe BA : AC � AC : CD, by divifion BC : CA � AD : DC. Again, becaufe BA : BC ꓰ BC : CE, converting and dividing BC : CA ꓰ CE : EB. Therefore, CE : EB ꓰ AD : DC. *Q. E. D.* S. HORSLEY.

LET AC, CB, be any two given right lines, and let CD in AC produced be equal to CB. On AD defcribe a femi-circle; draw CN at right angles to AD, and equal to CD (11. E. 1.); join A, N, and apply a right line AM in the femi-circle equal to AN (1. E. 4.). From the point M draw the right line MS at right angles to AD (1. E. 4.) Make a triangle ACB, having its fides equal to AC, AS, and CB (by 22. E. 1.); and ACB is the angle required to be found; and the fegments AL, LB, of the right line AB joining the extremities A and B of the given lines are refpectively equal to the fegments AP, BQ, of the given lines, which are adjacent to them. For the fquare on BC hath to the fquare on AC the duplicate proportion of BC to AC (cor. 1. to 20. E. 6.); that is, the proportion of BL to LA (prop. 1.). Wherefore the fquares on AC, CB; that is, the fquare on AN or AM (47. E. 1.) hath to the fquare on AC the proportion of AB to AL (18. E. 5.). But the fquare on AN or AM is equal to the rectangle contained by AD, AS (8. and 17. E. 6.). Wherefore the rectangle contained by AD, AS, hath to the fquare on AC, the proportion of AB to AL; that is, the proportion of the rectangle contained by AB, AS to the rectangle contained by AL, AS (1. E. 6.). Confequently, the proportion of the rectangle AD, AS, to the rectangle

rectangle AB, AS; that is, the proportion of AD to AB (1.
E. 6.) is equal to the proportion of the fquare on AC to
the rectangle AL, AS (16. E. 5.) But AD hath to AC the
proportion of AC to AP (cor. 3. to prop. 1.). Therefore,
the rectangle AD, AC, hath to the fquare on AC the propor-
tion of the rectangle AC, AS, to the rectangle AP, AS (1.
E. 6.). Therefore AD hath to AS or AB the proportion of
the fquare on AC to the rectangle AP, AS (16. E. 5. and
1. E. 6.).   Hence the rectangle AL, AS, is equal to the
rectangle AP, AS (11. and 16. E. 5.), and AL, AP, are con-
fequently equal.   But as AL to AP, fo is BL to BQ (by cor.
5. prop. 1.).   Therefore BL, BQ, are likewife equal.
Q. E. D.

PROPOSITION III. PROBLEM I.

To multiply the ſquare of a given finite right line by any number.

UPON an indefinite right line AP ſet off the given right line AB; draw BC at right angles to AP and equal to AB; and from A through C draw an indefinite right line AQ. Take AD equal to AC, and draw DE parallel to BC; AF equal to AE; and draw FG parallel to BC, and ſo on. Then it appears (from 47. E. I.), that the ſquare of AC

is

is equal to the fquare of AB multiplied by 2; the fquare
of AE equal to the fquare of AD or AC multiplied by 2;
that is, equal to the fquare of AB multiplied by 4, and fo
on. Thus the fquares of AC, AE, AG, AI, AL, &c. are re-
fpectively equal to the fquare of AB multiplied by the
terms of the following feries 2, 4, 8, 16, 32, 64, &c.
where the fixty-third term gives the fquare of AB mul-
tiplied by the laft term of SESSA's Series for the Chefs-
board.

If cx be drawn parallel to AP, the fquares of A*a*, A*b*,
A*c*, A*d*, &c. will be refpectively equal to the fquare of
AB multiplied by 3, 5, 9, 17, 33, 65, 129, &c. Alfo
if A*g* be taken equal to A*a*, and *ge* be drawn parallel to
BC, and this be repeated, the fquares of A*e*, &c. will be
equal refpectively to the fquare of AB multiplied by
6, 12, 24, 48, &c. And the fquares on A*o*, &c. will be
equal to fquare on AB multiplied by 4, 7, 13, 25, 49,
&c. In like manner, if AM be taken equal to A*b*, and MN
be drawn parallel to BC, the fquares on AN, &c. will be
equal refpectively to the fquare on AB multiplied by
10, 20, 40, 80, 160, &c. And the fquares on AS, &c.
will be equal refpectively to the fquare on AB multiplied
by the terms of the following feries: 6, 11, 21, 41, 81,
161, &c.

In the fame way, if right lines be drawn from E, *e*, G,
N, I, L, &c. there will arife numberlefs other feries. And
if BC be taken equal to AB multiplied by any number,
furd, fractional, or mixed, there will be obtained a great
variety of feries, confifting refpectively of terms, which
are

are furd, fractional, or mixed. And by dividing BC, DE, *g e*, FG, MN, HI, KL, in different ways, according to plea-fure, we may apply the fame method to fractional num-bers, without altering the magnitude of BC. Thus, if BC be bifected, and a right line be drawn through the point of bifection parallel to AP, there will be found right lines, the fquares on which are refpectively equal to the fquare on AB multiplied by a great number of fractions, having four for their common denominator, and fo on.

PROPOSITION IV. PROBLEM II.

To find a right line, the fquare on which fhall be equal to the fquare on a given right line, divided by any number.

IF, ufing the figure of the immediately preceding pro-blem, we fuppofe the given right line to be denoted by AL, the fquares on AK, AH, AF, AD, AB, &c. will re-fpectively be equal to the fquare on AL multiplied by $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, $\frac{1}{32}$, $\frac{1}{64}$, $\frac{1}{128}$, $\frac{1}{256}$, $\frac{1}{512}$, $\frac{1}{1024}$, &c. or divided by 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, &c.; and fo on for other numbers, whole, furd, fractional, or mixed.

PROPOSITION V. PROBLEM III.

To cut off from a given right line a part expreffed by any odd number.

LET AB be the given right line. At right angles to it, at one of its extremities B, draw an indefinite right line BE. Let n be the given odd number, which expreffes the part of AB which is to be cut off. Take BC fuch a right line (prob. 8.), that the fquare on it fhall be equal to the fquare on AB, multiplied by the number $\frac{n-1}{2}$. Draw CL as in the firft theorem, and take LS equal to LB. I fay, AS is that part of AB, which is expreffed by the odd number n.

For the fquare on AC being equal to the fquares on AB, BC, is equal to the fquare on AB multiplied by the number $\frac{n-1}{2} + 1$, or $\frac{n+1}{2}$. Therefore it appears (from prop. 1. and cor. 1. to 20. E. 6.), that AL is to LB as $\frac{n-1}{2} + 1$ to $\frac{n-1}{2}$. Confequently, AS is equal to the part required. *Q. E. F.*

3   Thus,

Thus, if the square on BC be supposed successively equal to the square on AB multiplied by the terms of the series 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, &c. the numbers of the several parts denoted by AS, will be 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35, 37, 39, 41, &c. which series comprehends all odd numbers after 9, and might have begun from 3 had the other series begun from 1.

P R O▬

PROPOSITION VI.   PROBLEM IV.

To cut off from a given right line a part exprefsed by any even number.
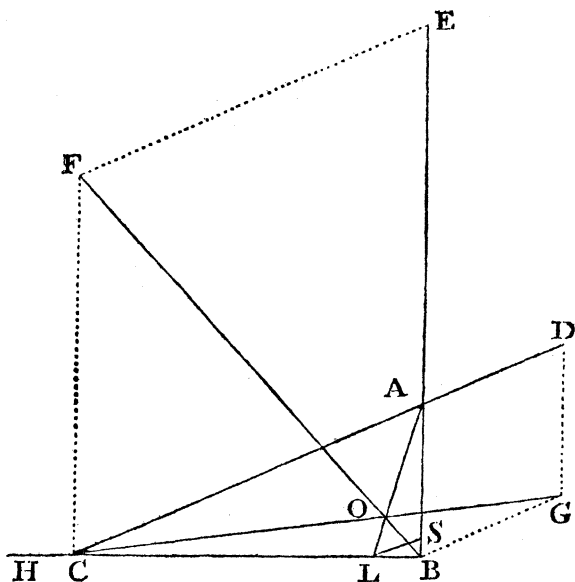
Let *m* denote any even number in general. Draw any indefinite right line BH, and at right angles to it another BE. On BE fet off the given right line BA, and from A, with the diftance equal to a right line, the fquare on which is equal to the fquare on AB multiplied by the number

number *m*−1, interfect BH in fome point c. From the
vertex A of the triangle BAC draw AL as was directed in
prop. 1. and draw LS parallel to CA. I fay BL is fuch a
part of BC as is expreffed by the number *m*; and that BS
is the fame part of AB. For it appears (from prop. 1.)
that BL is to CL as 1 to *m*−1. Wherefore BL is the *m*th
part of BC. And BS is the fame part of AB that BL is of
BC (4. E. 6.)

Thus if the fquare on AC be fucceffively denoted by
the fquare on AB multiplied by 3, 5, 7, 9, 11, 13, 15,
17, 19, 21, 23, 25, &c. BS will be fucceffively fuch a
part of AB as is expreffed by 4, 6, 8, 10, 12, 14, 16, 18,
20, 22, 24, 26, 28, &c.

PROPOSITION VII. THEOREM II.

*If from the angles at the bafe of any right lined triangle,
right lines be drawn to the alternate angles of rhombi
defcribed on the other two fides, and reciprocally applied to
them produced, and through the interfection of thefe lines,
a right line be drawn from the vertex to the bafe; the
rectangle contained by the fines of the angles at the ex-
tremities of one of the fides, will be equal to the rectangle
contained by the fines of the angles at the extremities of
the other (b); and the parallelepiped contained by the fines of
the*

(b) The author means, that $\overline{\text{fin. ACL} \times \text{fin. CAL}} = \text{fin. BCL} \times \text{fin. CBL}$ (fee
fig. prop. 1.). For fin. ACL : fin. L = AL : AC, and fin. L : fin. BCL = BC : BL. Take
N, a third in proportion to AL, AC. Then, becaufe $\text{AC}^2 : \text{BC}^2 = \text{AL} : \text{LB}$, N will

likewife

the angles of one of thofe triangles, into which the origi-
nal one is divided by the faid line drawn from the vertex.
will be equal to the parallelepiped contained by the fines of
the angles of the other.

COR. The two triangles, adjacent to the fegments of the bafe, have to each other the proportion of the two adjacent to the fides containing the vertical angle, or the proportion of the two into which the original triangle is divided; and any one of thefe pairs of triangles are as fimilar figures defcribed on the fides, being as the fegments of the bafe, which have to each other the duplicate proportion of the fides.

PROPOSITION VIII. THEOREM III.

If from the angles at the hypotenufe of any right angled
right lined triangle, right lines be drawn to the alternate
angles of fquares defcribed on the fides containing the
right angle, and from the point where the right line
drawn from the right angle, through their interfection,
meets the hypotenufe, right lines be drawn to the points,
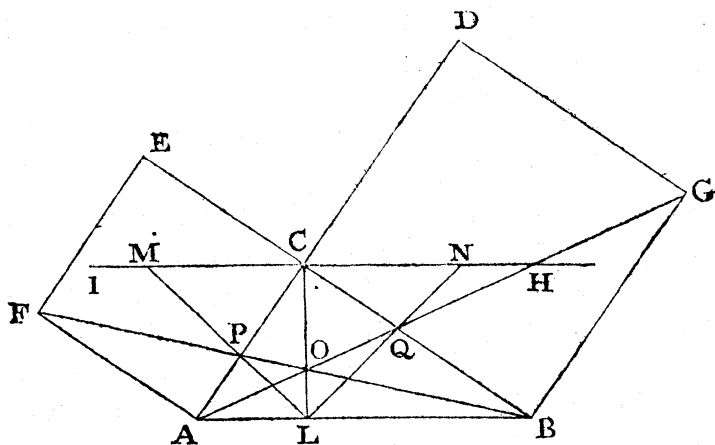*where thefe lines meet the fides; the lines fo drawn will*

likewife be a third in proportion to LB, BC. Hence, fin. ACL : fin. L $=$ AC : N, and fin. L : fin. BCL $=$ N : BC. *Ex æquo perturbate* fin. ACL : fin. BCL $=$ AC : BC. But fin. CBL : fin. CAL $=$ AC : BC. Therefore, fin. ACL : fin. BCL $=$ CBL : fin. CAL, and $\overline{\text{fin. ACL} \times \text{fin. CAL}} = \overline{\text{fin. BCL} \times \text{fin. CBL}}$. *Q. E. D.* And hence, fin. L $\times$ fin. ACL $\times$ fin. CAL $=$ fin. L $\times$ fin. BCL $\times$ fin. CBL, which is the fecond branch of the propofition. S. HORSLEY.

make

*make equal angles with the hypotenuse, and the right line
drawn from the right angle to meet it; and will likewise
have to each other the proportion of the sides containing
the right angle* (c).

COR.

(c) For in the triangle ACB, having the angle at c right, conftruct the figure
as in the firft propofition, and join LP, LQ. The author affirms, that the angles

PLA, QLB, and alfo the angles CLP, QLC, are equal, and that PL : LQ=CA : CB.
Produce LP, LQ, till they meet the right line IH in M, N. Now, becaufe MN, AB,
are parallel, therefore AL : MC=AP : PC, and CN : LB = CQ (or PC) : QB.
Therefore, AL × CN : MC × LB=AP : QB. But AP : QB=AL : LB (by theor. 1.
cor. 5.). Therefore, AL × CN : MC × LB=AL : LB. Therefore, CN and CM are
equal. But becaufe the angle ACB is right, CL is perpendicular to AB (by
theor. 1. cor. 2.), and confequently to MN. Therefore the angles MCL, NCL,
are right, and MC being equal to CN, and CL common to the two triangles MCL,
NCL, ML will be equal to LN, and the angles MLC, CML, equal to NLC, CNL, refpec-
tively. Hence it is evident, that the angles CLP, CLQ, are equal, and each is
half a right angle; and likewife PLA, QLB, are equal, and each half a right angle.
                                                                    Further,

COR. I. The alternate triangles of thoſe four, which have their vertices in the point, where the right line drawn from the right angle meets the hypotenuſe, are ſimilar, and have to each other the proportion of the ſegments of the hypotenuſe, or the duplicate proportion of the ſides containing the right angle.

COR. II. Either pair of the adjacent triangles lying on different ſiles of the right line drawn from the right angle, and having their vertices in the interſection of the right lines drawn from the angles at the hypotenuſe, have to each other the proportion of the alternate triangles, having their vertices in the interſection of the firſt-mentioned line and the hypotenuſe.

COR. III. The trapezium or quadrilateral figure formed by the ſegments of the ſides adjacent to the right angle, and the right lines joining their extremities with the interſection of the hypotenuſe, and the right line drawn from the right angle to meet it, is capable of being inſcribed in a circle; and is divided at the interſection of right lines drawn from the angles at the hypotenuſe to the alternate angles of ſquares, deſcribed on the ſides containing the right angle, into triangles which are proportional to one another, and when taken two by two, as

Further, the angles, PLA, CLQ, are equal, each being half a right angle. And the angles PAL, LCQ, are equal (by 8. Elem. 6.). Therefore the angles APL, CQL, will be equal, and the triangles APL, CQL, ſimilar, and the ſides ſubtending the equal angles proportional. Therefore, PL : LA $=$ QL : LC. By permutation, LP : LQ $=$ LA : LC. But LA : LC $=$ AC : CB (by 8. Elem. 6.). Therefore, LP : LQ $=$ AC : CB. *Q. E. D.* S. HORSLEY.

they

they lie adjacent on different fides of the diagonal, are proportional to the unequal fides of the trapezium, and to the two triangles into which the diagonal divides it. For OPL : OPC=LO : OC=OQL : OQC. Therefore, OPL : OQL =OPC : OQC=LPC : LQC=LP : LQ.

PROPOSITION IX. THEOREM IV.

If from the angles at the bafe of any right lined triangle, right lines be drawn to the alternate angles of rhomboids defcribed on the other two fides, and reciprocally applied to them produced, a right line drawn from the vertex through the interfection of thefe lines will cut the bafe into two parts, having to each other the proportion compounded of the proportion of the fides, and of the proportion of the other two lines comprehending the rhomboids.

I SHALL fubjoin a demonftration of this theorem, fince the firft propofition in thefe papers is only a particular cafe of it.
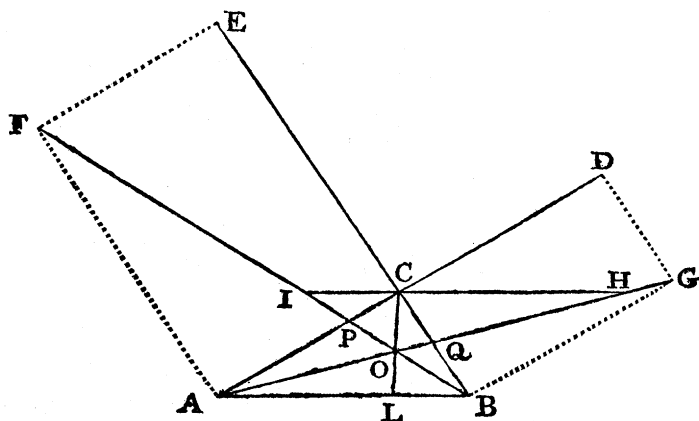
Let the triangle be ACB, the bafe AB, the rhomboids
ACEF, CCDG; and let the right lines BF, AG, be drawn.
Then, if from the vertex C through their interfection O,
a right line COL be drawn to meet the bafe, the fegments
AL, LB, thereof will have to each other the proportion
compounded of the proportions of AC to CB, and of CE
to CD.   For through the vertex C let a right line ICH be
drawn parallel to AB, to meet BF, AG, produced, if necef-
fary. Then, fince the triangles CQH, CPI, are refpectively
equiangular to the triangles AQB, APB (15. and 29. E. 1.);
the proportions of CH to AB, and of AB to IC are refpec-
tively equal to the proportions of CQ to QB, and of AP to
PC (4. E. 6.). But the proportion of CH to IC is com-
pounded of the proportions of CH to AB and of AB to IC;
and confequently is equal to a proportion compounded of
the proportions of CQ to QB and of AP to PC. And fince
the triangles ACQ, APF, are refpectively equiangular to

I   the

the triangles BQG, BPC (15. and 29. E. 1.); the propor-
tions of CQ to QB, and of AP to PC, are refpectively equal
to the proportions of AC to BG or CD, and of AE or CE to
CB. Wherefore the proportion of CH to IC is equal to the
proportion compounded of the proportions of AC to CD
and of CE to CB, or of AC to CB and of CE to CD. But
fince the triangles COH, COI, are refpectively equiangular
to the triangles AOL, LOB, the proportion of CH to IC is
equal to the proportion of AL to LB (4. E. 6.). There-
fore the proportion of AL to LB is equal to the propor-
tion compounded of the proportions of AC to CB and of
CE to CD. *Q. E. D.*

SCHOLIUM. If CE, CD, be equal to each other, AL hath
to LB the proportion of AC to CB, and CL bifects the an-
gle ACB; if CE have to CD the inverfe proportion of AC to
CB, AL is equal to LB; if CE have to CD the proportion of
AC to CB, AL hath to LB the duplicate proportion of AC to
CB; and univerfally, if CE have to CD any multiplicate
proportion, *n*, of AC to CB, AL hath to LB fuch a multipli-
cate proportion of AC to CB as is expreffed by the num-
ber *n*+1. And if CE have to CD any multiplicate pro-
portion *m* of CB to AC, AL will have to LB fuch a multi-
plicate proportion of CB to AC, as is expreffed by the num-
ber *m*−1.

IV. *A*